COMBINATORICA 4 (1) (1984) 61—70

ON RANDOM MAPPING PATTERNS

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Dedicated to Paul Erdős on his seventieth birthday

Received 13 February 1983

Random mapping patterns may be represented by unlabelled directed graphs in which each point has out-degree one. We determine the asymptotic behaviour of various parameters associated with such graphs, such as the expected number of points belonging to cycles and the expected number of components.

1. Introduction

Let f denote a function that maps the set $N=\{1, 2, ..., n\}$ into itself. The function f may be represented by a directed graph D with point-set N and arc-set $\{(i, f(i)) \colon i \in N\}$; such graphs, in which every point has out-degree one, are sometimes called *functional digraphs* [4; p. 69]. Two such functions f and g are equivalent if there exists a permutation π of N such that $f(i)=j$ if and only if $g(\pi(i))=\pi(j)$ for all $i \in N$. An equivalence class of mapping functions is called a *mapping pattern.* Let p_n denote the number of mapping patterns on n points or, equivalently, the number of functional digraphs with n unlabelled points.

De Bruijn and Klarner [1; p. 367] give some historical comments on the papers that have dealt with the problem of determining the numbers p_n. In particular, Harary [3] applied Pólya's theorem to obtain a formal expression for the generating function $P(x)=1+\sum_1^\infty p_n x^n$ in terms of the generating function $T(x)=\sum_1^\infty t_n x^n$, where t_n denotes the number of rooted trees with n unlabelled points. Read [11] simplified this expression and found that $P(x)=\prod_1^\infty (1-T(x^j))^{-1}$; de Bruijn and Klarner [1] gave a combinatiorial derivation of Read's relation.

Our main object here is to determine the asymptotic behaviour of various parameters associated with random mapping patterns and their corresponding graphs. In Section 2 we summarize some results that we shall need later; in particular, we list some properties of the function $T(x)$. In Section 3 we determine, asymptotically, the number of mapping patterns whose graphs are connected. These graphs consists of a collection of rooted directed trees whose roots coincide with the points of a

AMS subject classification (1980): 05 C 30; 05 C 20

directed cycle; the arcs of each tree are directed towards the root of the tree. We also determine the expected number of points in the cycles in such graphs. In Section 4 we determine the asymptotic behaviour of p_n. We also determine the expected number of points belonging to cycles and the expected number of components in the graphs of random mapping patterns. Finally, in Section 5 we compare these results for random mapping patterns with the corresponding results obtained earlier by others (see. e.g. [10; §7.1]) for random mapping functions in which the labels of the points are taken into account.

2. Preliminary results

We summarize here some results we shall need later. The first part of the following lemma is due to Darboux [2; p. 20]; the second part follows from an extension of Darboux's result due to Jungen [8; p. 275].

Lemma 1. *Suppose the function* $g(x)=\sum_0^\infty g_n x^n$ *has a non-zero finite radius of convergence* ϱ *and that* $x=\varrho$ *is the only singularity on the circle of convergence.*

(i) *If* $g(x)$ *has an expansion about* $x=\varrho$ *of the form*

$$g(x) = (\varrho-x)^{-s}A(x)+B(x),$$

where $A(x)$ *and* $B(x)$ *are analytic for* $|x|\leqq\varrho$, $A(\varrho)\neq 0$ *and* $s\neq 0, -1, -2, \ldots,$ *then*

$$g_n = \frac{A(\varrho)}{\Gamma(s)}\varrho^{-n-s}n^{s-1}+O(\varrho^{-n}n^{s-2})$$

as $n\to\infty$.

(ii) *If* $g(x)$ *has an expansion about* $x=\varrho$ *of the form*

$$g(x) = (\varrho-x)^{-s}\log(\varrho-x)+(\varrho-x)^{-s}A(x)+B(x),$$

where $A(x)$ *and* $B(x)$ *are analytic for* $|x|\leqq\varrho$ *and* $s\neq 0, -1, -2, \ldots,$ *then*

$$g_n = \frac{1}{\Gamma(s)}\varrho^{-n-s}n^{s-1}\log n+O(\varrho^{-n}n^{s-1})$$

as $n\to\infty$. ∎

Otter [14] (see also [4; §9.5]) showed that the generating function $T(x) = \sum_1^\infty t_n x^n$ for the rooted unlabelled trees has radius of convergence $\varrho=.3383\ldots$, that $x=\varrho$ is the only singularity on the circle of convergence, and that $T(x)$ has an expansion about $x=\varrho$ of the form

$$T(x) = 1-b(\varrho-x)^{1/2}+b_2(\varrho-x)+b_3(\varrho-x)^{3/2}+\ldots \tag{2.1}$$

where $b=2.6811\ldots$; from this he deduced that

$$t_n = \frac{1}{2}b(\varrho/\pi)^{1/2}\varrho^{-n}n^{-3/2}+O(\varrho^{-n}n^{-5/2})$$

where $\frac{1}{2}b(\varrho/\pi)^{1/2}=.4399\ldots$.

The following properties of the function $T(x)$ are straightforward consequences of Otter's results.

Lemma 2. (i) *$T(x)$ is analytic for $|x| \leqq \varrho$, $x \neq \varrho$; furthermore, $|T(x)| < 1$ if $|x| \leqq \varrho$, $x \neq \varrho$.*

(ii) *$|T(x^j)| \leqq T(\varrho^j)$ for $|x| \leqq \varrho$ and $j = 2, 3, \ldots$.*

(iii) *$T(x^j)$ is analytic for $|x| < \varrho^{1/2}$ and $j = 2, 3, \ldots$.*

(iv) *If $|x| \leqq \varrho$ then $|T(x)/x| \leqq T(\varrho)/\varrho = 1/\varrho$; in particular, $T(\varrho^j) \leqq \varrho^{j-1}$ for $j = 1, 2, \ldots$.* ∎

3. On connected mapping patterns

If $F(x)$ denotes any power series, let $Z\{C_k, F(x)\}$ and $Z\{S_m, F(x)\}$ denote the power series obtained by substituting $F(x)$ into the cycle index of the cyclic group C_k and the symmetric group S_m, respectively. We recall that

$$Z\{C_k, F(x)\} = \frac{1}{k} \sum_{d|k} \varphi(d)\big(F(x^d)\big)^{k/d}, \tag{3.1}$$

where φ denotes the Euler phi-function, and that

$$1 + \sum_{m=1}^{\infty} Z\{S_m, F(x)\} z^m = \exp \sum_{1}^{\infty} z^m F(x^m)/m \tag{3.2}$$

for any variable z. (For the definition of the cycle index of a permutation group and an exposition of the role of cycle-indices in Pólya's enumeration theorem, see, e.g., [4; Chap. 2]).

Let c_n denote the number of connected mapping patterns on n points, that is, the number of mapping patterns on n points whose graphs are connected; let $C(x) = \sum_{1}^{\infty} c_n x^n = x + 2x^2 + \ldots$ denote the generating function for the numbers c_n.

Theorem 1. $C(x) = -\sum_{1}^{\infty} \frac{\varphi(d)}{d} \log\big(1 - T(x^d)\big)$.

Proof. We remarked earlier that the graph of a connected mapping pattern consists of a collection of rooted unlabelled directed trees whose roots coincide with the points of a directed cycle (the arcs of each tree are directed towards the root of the tree). Harary [3] observed that it therefore follows immediately from Pólya's theorem that the generating function for connected mapping patterns in which the cycle has length k is $Z\{C_k, T(x)\}$ and that, consequently,

$$C(x) = \sum_{1}^{\infty} Z\{C_k, T(x)\}.$$

Read [11] simplifed this sum by appealing to equation (3.1) and interchanging the order of summation; this gives rise to the relation

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{d|k} \varphi(d)(T(x^d))^{k/d}$$

$$= \sum_{d=1}^{\infty} \frac{\varphi(d)}{d} \sum_{j=1}^{\infty} \frac{1}{j} (T(x^d))^j = -\sum_{d=1}^{\infty} \frac{\varphi(d)}{d} \log(1-T(x^d)). \quad \blacksquare$$

Corollary 1.1. $c_n = \varrho^{-n}/2n + O(\varrho^{-n} n^{-3/2})$.

Proof. For notational convenience we let $\mathscr{C}_n\{g(x)\}$ denote the coefficient of x^n in the power series $g(x)$.

It follows from Lemma 2(i) that the function $\log(1-T(x))$ is analytic for $|x| \leqq \varrho$, $x \neq \varrho$; around $x = \varrho$ it has the expansion

$$\log(1-T(x)) = \frac{1}{2} \log(\varrho - x) + \log(b - b_2(\varrho-x)^{1/2} - \dots)$$

$$= \frac{1}{2} \log(\varrho - x) + \alpha_0 + \alpha_1(\varrho - x)^{1/2} + \alpha_2(\varrho - x) + \dots$$

for suitable constants $\alpha_0, \alpha_1, \dots,$ in view of (2.1). Now $\mathscr{C}_n(\log(\varrho - x)) = -\varrho^{-n}/n$, and when we apply Lemma 1(i) to the remaining terms in the expansion we find that

$$\mathscr{C}_n\{-\log(1-T(x))\} = \varrho^{-n}/2n + O(\varrho^{-n} n^{-3/2}).$$

Let ε denote an arbitrary fixed positive constant; then there exists a constant K such that if $|z| \leqq 1-\varepsilon$ then $|\log(1-z)| \leqq K|z|$. Thus it follows from Lemma 2 (iv) that $|\log(1-T(x^d))| \leqq K|T(x^d)| \leqq K\varrho^{d-1}$ for $d = 2, 3, \dots,$ provided that $|T(x^d)| \leqq 1-\varepsilon$. This implies that the function $D(x) = -\sum_{2}^{\infty} \varphi(d) \log(1-T(x^d))/d$ converges uniformly for $|x| < \varrho + 2\eta < \varrho^{1/2}$ for a suitable fixed $\eta > 0$; consequently, $D(x)$ is analytic for $|x| < \varrho + 2\eta$ and, hence,

$$\mathscr{C}_n\{D(x)\} = O((\varrho+\eta)^{-n}) = O(\varrho^{-n} n^{-3/2}).$$

Therefore,

$$c_n = \mathscr{C}_n\{C(x)\} = \mathscr{C}_n\{-\log(1-T(x)) + D(x)\}$$

$$= \varrho^{-n}/2n + O(\varrho^{-n} n^{-3/2}). \quad \blacksquare$$

Let μ_n denote the expected length of the cycle in the graph of a random connected mapping pattern on n points, where the expectation is over all the c_n such patterns; let $M(x) = \sum_{1}^{\infty} \mu_n c_n x^n = x + 3x^2 + \dots$ denote the generating function for the numbers $\mu_n c_n$.

Theorem 2. $M(x) = \sum_{1}^{\infty} \varphi(d) T(x^d)(1-T(x^d))^{-1}$.

Proof. Since $Z\{C_k, T(x)\}$ is the generating function for connected mapping patterns in which the cycle has length k, it follows from (3.1) that

$$M(x) = \sum_{k=1}^{\infty} kZ\{C_k, T(x)\} = \sum_{k=1}^{\infty} \sum_{d|k} \varphi(d)(T(x^d))^{k/d}$$

$$= \sum_{d=1}^{\infty} \varphi(d) \sum_{j=1}^{\infty} (T(x^d))^j = \sum_{d=1}^{\infty} \varphi(d) T(x^d)(1-T(x^d))^{-1}. \quad \blacksquare$$

Corollary 2.1. $\mu_n = \frac{2}{b}(n/\pi\varrho)^{1/2} + O(1)$.

Proof. It follows from Lemma 2(i) that the function $T(x)(1-T(x))^{-1}$ is analytic for $|x| \leqq \varrho$, $x \neq \varrho$; around $x=\varrho$ it has the expansion

$$T(x)(1-T(x))^{-1} = \frac{1}{b}(\varrho-x)^{-1/2} + \beta_0 + \beta_1(\varrho-x)^{1/2} + \dots$$

for suitable constants $\beta_0, \beta_1, \dots$, in view of (2.1). Hence,

$$\mathscr{C}_n\{T(x)(1-T(x))^{-1}\} = \frac{1}{b}\varrho^{-n}(\pi\varrho n)^{-1/2} + O(\varrho^{-n} n^{-3/2}),$$

by Lemma 2(i). An argument similar to the one applied to the function $D(x)$ in the proof of Corollary 1.1 can be used to show that the function $V(x) = \sum_{2}^{\infty} \varphi(d) T(x^d) \cdot$ $\cdot (1-T(x^d))^{-1}$ is analytic for $|x| < \varrho + 2\eta < \varrho^{1/2}$, for some $\eta > 0$; hence

$$\mathscr{C}_n\{V(x)\} = O((\varrho+\eta)^{-n}) = O(\varrho^{-n} n^{-3/2})$$

and, consequently,

$$\mu_n c_n = \mathscr{C}\{M(x)\} = \mathscr{C}_n\{T(x)(1-T(x))^{-1} + V(x)\}$$

$$= \frac{1}{b}(\pi\varrho n)^{-1/2} + O(\varrho^{-n} n^{-3/2}).$$

This and Corollary 1.1 imply the required result. $\blacksquare$

4. On general mapping patterns

Let $P(x) = 1 + \sum_{1}^{\infty} p_n x^n = 1 + x + 3x^2 + \dots$, where p_n denotes the total number of mapping patterns on n points. The following expression for $P(x)$ in terms of $T(x)$ is due to Read [11].

Theorem 3. $P(x) = \prod_{1}^{\infty} (1-T(x^j))^{-1}$.

Proof. Harary [3] observed that it follows immediately from Pólya's theorem that the generating function for mapping patterns whose graph has m components is $Z\{S_m, C(x)\}$; hence,

$$P(x) = 1+\sum_1^\infty Z\{S_m, C(x)\} = \exp \sum_1^\infty \frac{1}{m} C(x^m) \tag{4.1}$$

in view of relation (3.2). Read [11] simplified this by appealing to Theorem 1, in effect, and interchanging the order of summation; this gives rise to the relation

$$P(x) = \exp(-\sum_{m=1}^\infty \frac{1}{m} \sum_{d=1}^\infty \frac{\varphi(d)}{d} \log(1-T(x^{dm})))$$

$$= \exp(-\sum_{j=1}^\infty \frac{1}{j} \log(1-T(x^j)) \sum_{d|j} \varphi(d)) = \prod_{j=1}^\infty (1-T(x^j))^{-1},$$

since $\sum_{d|j} \varphi(d)=j$. ∎

For notational convenience let $P^*(x)=\prod_2^\infty (1-T(x^j))^{-1}$ so that $P(x) =(1-T(x))^{-1}P^*(x)$.

Corollary 3.1. $p_n=\frac{P^*(\varrho)}{b} \varrho^{-n}(\varrho\pi n)^{-1/2}+(\varrho^{-n}n^{-3/2})$.

Proof. It follows from Lemma 2 (iii) and (iv) that the function $P^*(x)$ is analytic for $|x|<\varrho+\eta$ for some $\eta>0$. Thus $P(x)$ is analytic for $|x|\leqq\varrho$, $x\neq\varrho$; and around $x=\varrho$ it has the expansion

$$P(x) = \frac{P^*(\varrho)}{b} (\varrho-x)^{-1/2}+\delta_0+\delta_1(\varrho-x)^{1/2}+\dots$$

for suitable constants $\delta_0, \delta_1, \dots$, in view of (2.1). Hence

$$p_n = \mathscr{C}_n\{P(x)\} = \frac{P^*(\varrho)}{b} \varrho^{-n}(\varrho\pi n)^{-1/2}+O(\varrho^{-n}n^{-3/2}),$$

by Lemma 1 (i). ∎

Let r_{nk} denote the number of mapping patterns on n points whose graphs have k points belonging to cycles; let $R(x, y)=1+\sum_1^\infty r_{nk}x^n y^k$.

Theorem 4. $R(x, y)=\prod_1^\infty (1-y^jT(x^j))^{-1}$.

Proof. Let $R_1(x, y)$ denote the generating function in which the coefficient of $x^n y^k$ is the number of connected mapping patterns on n points whose graphs contain a cycle of k points. It follows by a slight extension of the argument used to prove Theorems 1 and 2 that

$$R_1(x, y) = \sum_1^\infty y^k Z\{C_k, T(x)\} = -\sum_1^\infty \frac{\varphi(d)}{d} \log(1-y^dT(x^d))$$

and that

$$R(x, y) = 1+\sum_1^\infty Z\{S_m, R_1(x, y)\}$$

$$= \exp \sum_1^\infty \frac{1}{m} R_1(x^m, y^m) = \prod_1^\infty (1-y^j T(x^j))^{-1}. \quad ▮$$

Let γ_n denote the expected number of points in cycles in the graph of a random mapping pattern on n points, where the expectation is over the p_n such patterns; let $G(x)=\sum_1^\infty \gamma_n p_n x^n = x+5x^2+\dots$ denote the generating function for the numbers $\gamma_n p_n$.

Corollary 4.1. $G(x)=P(x)\sum_1^\infty jT(x^j)(1-T(x^j))^{-1}$.

Proof. This follows upon taking the logarithmic derivative with respect to y of both sides of the relation $R(x, y)=\prod(1-y^jT(x^j))^{-1}$, setting $y=1$, and then appealing to the formula for $P(x)$ given in Theorem 3. ▮

Corollary 4.2. $\gamma_n=\frac{1}{b}(\pi n/\varrho)^{1/2}+O(1)$.

Proof. It can be shown, by the same type of argument that has been used earlier, that the function $U(x)=\sum_2^\infty jT(x^j)(1-T(x^j))^{-1}$ is analytic for $|x|<\varrho+\eta$ for some $\eta>0$. Thus it follows that the function

$$\begin{aligned} G(x) &= P(x)\{T(x)(1-T(x))^{-1}+U(x)\} \\ &= P^*(x)T(X)(1-T(x))^{-2}+P^*(x)U(x)T(x)(1-T(x))^{-1} \end{aligned}$$

is analytic for $|x|\leqq\varrho$, $x\neq\varrho$; and around $x=\varrho$ it has the expansion

$$G(x) = \frac{P^*(\varrho)}{b^2}(\varrho-x)^{-1}+\tau_1(\varrho-x)^{-1/2}+\tau_2+\tau_3(\varrho-x)^{1/2}+\dots$$

for suitable constants $\tau_1, \tau_2, \dots,$ in view of (2.1). Now $\mathscr{C}_n\{(\varrho-x)^{-1}\}=\varrho^{-n-1}$ and when we apply Lemma 1(i) to the remaining terms in the expansion we find that

$$\gamma_n p_n = \mathscr{C}_n\{G(x)\} = \frac{P^*(\varrho)}{b^2}\varrho^{-n-1}+O(\varrho^{-n}n^{-1/2}).$$

This and Corollary 3.1 imply the required result. ▮

Let h_{nm} denote the number of random mapping patterns on n points whose graphs have m components; let $H(x, y)=1+\sum_1^\infty h_{nm}x^n y^m$.

Theorem 5. $H(x, y)=\exp\sum_1^\infty y^m C(x^m)/m$.

Proof. We saw in the proof of Theorem 3 that $Z\{S_m, C(x)\}$ is the generating function for mapping patterns whose graphs have m components. It follows, therefore, that

$$H(x, y) = 1 + \sum_1^\infty y^m Z\{S_m, C(x)\} = \exp \sum_1^\infty y^m C(x^m)/m,$$

upon appealing to the definition of $H(x, y)$ and relation (3.2). ∎

Let λ_n denote the expected number of components in the graph of a random mapping pattern on n points, where the expectation is over the p_n such patterns; let $L(x) = \sum_1^\infty \lambda_n p_n x^n = x + 4x^2 + \dots$ denote the generating function for the numbers $\lambda_n p_n$.

Corollary 5.1. $L(x) = P(x) \sum_1^\infty C(x^m)$.

Proof. This follows readily upon taking the derivative with respect to y of both sides of the formula for $H(x, y)$ in Theorem 5, setting $y=1$, and then appealing to relation (4.1). ∎

Corollary 5.2. $\lambda_n = \frac{1}{2} \log n + O(1)$.

Proof. We saw earlier, in the proof of Corollary 1.1, that $\mathscr{C}(x) = -\log(1-T(x)) + D(x)$ where $D(x)$ is analytic for $|x| < \varrho + 2\eta$ for a suitable $\eta > 0$. It is not difficult to see that the function $W(x) = \sum_2^\infty C(x^m)$ is also analytic for $|x| < \varrho + 2\eta$, if $\eta > 0$ is small enough. Thus the function

$$\begin{aligned} L(x) &= P(x)\{-\log(1-T(x)) + D(x) + W(x)\} \\ &= -P^*(x)(1-T(x))^{-1} \log(1-T(x)) + P^*(x)(D(x)+W(x))(1-T(x))^{-1} \end{aligned}$$

is analytic for $|x| \leqq \varrho$, $x \neq \varrho$; and around $x = \varrho$ it has the expansion

$$L(x) = -\frac{P^*(\varrho)}{2b}(\varrho - x)^{-1/2} \log(\varrho - x) + \nu_1(\varrho - x)^{-1/2} + \nu_2 + \dots$$

for suitable constants $\nu_1, \nu_2, \dots$, in view of (2.1). Hence,

$$\lambda_n p_n = \mathscr{C}_n\{L(x)\} = \frac{P^*(\varrho)}{2b} \varrho^{-n} (\pi \varrho n)^{-1/2} \log n + O(\varrho^{-n} n^{-1/2})$$

by Lemma 1 (ii). This and Corollary 3.1 imply the required result. ∎

We remark that the foregoing problems can also be considered for mapping patterns with no fixed points, i.e., patterns whose graphs have no cycles of length 1. In particular, let $Q(x) = 1 + \sum_1^\infty q_n x^n$ denote the generating function for mapping patterns with no fixed points; Read [11] showed that $Q(x) = x(T(x))^{-1} \cdot \prod_1^\infty (1 - T(x^j))^{-1} = xP(x)/T(x)$. It follows readily from this relation that $q_n \sim \varrho p_n$ as $n \to \infty$.

5. Concluding remarks

The coefficients in the generating function $T(x)=\sum_1^\infty t_n x^n$ are bounded above by the corresponding coefficients in the generating function

$$y(x) = \sum \frac{1}{n}\binom{2n-2}{n-1}x^n = \frac{1}{2}(1-(1-4x)^{1/2});$$

see [14; p. 590] or [4; p. 209]. This enables one to estimate $T(\varrho^j)$ for $j=2, 3, \ldots$; the case $j=2$ is illustrated in [9; p. 326]. If the product

$$P^*(\varrho) = \prod_2^\infty (1-T(\varrho^j))^{-1}$$

is estimated by the factors corresponding to $j=2, \ldots, K-1$, the truncation error can be estimated from the inequalities

$$1 < \prod_K^\infty (1-T(\varrho^j))^{-1} < \exp\left\{(1-y(\varrho^K))^{-1}\sum_K^\infty T(\varrho^j)\right\} <$$
$$< \exp\left\{y(\varrho^K)((1-\varrho)(1-y(\varrho^K)))^{-1}\right\}.$$

In this way we find that $P^*(\varrho)=1.2241\ldots$.

It follows from Corollary 3.1 that the number of mapping patterns on n points is $\sim(.442\ldots)\varrho^{-n}n^{-1/2}$. Thus the probability that the graph of a random mapping pattern on n points is connected is $\sim(1.128\ldots)n^{-1/2}$, by Corollary 1.1. Katz [6] and Rényi [12] have shown that the probability that the graph of a random mapping function on n labelled points is connected is $\sum_1^n (n)_k n^{-k}\sim(\pi/2n)^{1/2}=(1.125\ldots)n^{-1/2}$.

The expected length of the cycle in the graph of a random connected mapping pattern on n points is $\sim(.723\ldots)n^{1/2}$ by Corollary 2.1. Rényi [12] has shown that the corresponding expectation for the graph of a random connected mapping function on n labelled points is $\sim(.797\ldots)n^{1/2}$; in fact, the expected length of the cycle equals the reciprocal of the probability that the graph of a random mapping function is connected.

The expected number of points in cycles in the graph of a random mapping pattern on n points is $\sim(1.136\ldots)n^{1/2}$, by Corollary 4.2. It follows from a result proved by Harris [5] and Riordan [13] that the corresponding expectation for the graph of a random mapping function on n labelled points is $(1.125\ldots)n^{1/2}$; in fact, the expected number of points in cycles equals n times the probability that the graph of a random mapping function is connected.

The expected number of components in the graph of a random mapping pattern on n points is $\sim \frac{1}{2}\log n$, by Corollary 5.2. Kruskal [7] (see also Riordan [13]) has shown that the corresponding expectation for the graph of a random mapping function on n labelled points is also $\sim \frac{1}{2}\log n$.

Acknowledgement. The preparation of this paper was assisted by grants from the Natural Sciences and Engineering Research Council of Canada.

Note added in in proof. One of the referees has pointed out that additional results on random mapping functions on labelled point-sets are contained in V. Stepanov, Limit distributions of certain characteristics of random mappings, *Theory Probability Appl.*, **14** (1969), 612—626 and Yu. L. Pavlov, Limit distributions of certain characteristics of random mappings, *Theory Probability Appl.*, **26** (1981), 829—834.

References

[1] N. G. de Bruijn and D. A. Klarner, Multisets of aperiodic cycles, *SIAM J. Alg. Disc. Math.* **3** (1982), 359—368.
[2] G. Darboux, Mémoire sur l'approximation des fonctions de très grands nombres, et sur une classe étendu développements en série, *J. Math. Pures et Appliquées* **4** (1878), 5—56.
[3] F. Harary, The number of functional digraphs, *Math. Ann.* **138** (1959), 203—210.
[4] F. Harary and E. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.
[5] B. Harris, Probability distributions related to random mappings, *Ann. Math. Statist.* **31** (1960), 1045—1062.
[6] L. Katz, Probability of indecomposability of a random mapping function, *Ann. Math. Statist.* **26** (1955), 512—517.
[7] M. D. Kruskal, The expected number of components under a random mapping function, Amer. Math. Monthly **61** (1954), 392—397.
[8] R. Jungen, Sur les séries de Taylor n'ayant que des singularités algébricologarithmiques sur leur cercle de convergence, *Comm. Math. Hel.* **3** (1931), 266—306.
[9] A. Meir and J. W. Moon, Path edge-covering constants for certain families of trees, *Util. Math.* **14** (1978), 313—333.
[10] J. W. Moon, Counting Labelled Trees, *Canadian Mathematical Congress*, Montreal, 1970.
[11] R. C. Read, A note on the number of functional digraphs, *Math. Ann.* **143** (1961), 109—110.
[12] A. Rényi, On connected graphs, I, *Publ. Math. Inst. Hung. Acad. Sci.* **4** (1959), 385—388.
[13] J. Riordan, Enumeration of linear graphs for mappings of finite sets, *Ann. Math. Statist.* **33** (1962), 178—185.
[14] R. Otter, The number of trees, *Ann. of Math.* **49** (1948), 583—599.

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